

The Stern diatomic sequence via generalized Chebyshev polynomials

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November 17, 2015

The Stern diatomic sequence is defined by

$$a(0) = 0, \quad a(1) = 1, \quad a(2n) = a(n), \quad a(2n+1) = a(n) + a(n+1). \quad (1)$$

See [4] for a nice survey article.

Define polynomials $q_r(y_1, \dots, y_r)$ inductively by

$$\begin{aligned} q_0 &= 1, \quad q_1(y_1) = y_1, \\ q_r(y_1, \dots, y_r) &= y_1 q_{r-1}(y_2, \dots, y_r) - q_{r-2}(y_3, \dots, y_r) \text{ for } r \geq 2. \end{aligned} \quad (2)$$

The main result of this note is that $a(n)$ coincides with these polynomials when the variables are given the value of the gaps between successive 1's in the binary expansion of n , increased by 1 (Theorem 1), and a formula expressing the same polynomials in terms of sets of increasing integers of alternating parity (Theorem 2).

The polynomials q_r have appeared before as generalized Chebyshev polynomials in the context of cluster algebras, see the paper [3, Lemma 3.2].

A special case coincides with a recent expression obtained by Defant [2, eq (29)]. Using the polynomial representation, we give simple proofs to a number of known identities for $a(n)$, including a convolution identity found by Coons [1] (Corollary 3), and we derive a result on the divisibility of $a(n)$ (Corollary 4).

Let $c_i \geq 0$ be non-negative integers, and define $d_i = \sum_{j=1}^i c_j$, $[c_1, \dots, c_r] = 2^{d_r} + 2^{d_{r-1}} + \dots + 2^{d_1} + 1$. Note that if $c_i \geq 1$, then c_i is the distance between the two consecutive 1's corresponding to $2^{d_{i-1}}$ and 2^{d_i} in the binary expansion of the odd integer $n = [c_1, \dots, c_r]$. In this case, clearly $r+1 = s(n)$, the sum of the digits of the binary expansion of n . However, in general $2^{d_r} + 2^{d_{r-1}} + \dots + 2^{d_1} + 1$ is not necessarily the binary expansion of n . If $r < i$, we define $[c_i, \dots, c_r] = 1$.

Formula (3) below is proved by induction on c , and the others follow easily from the definitions.

$$a(2^c n + 1) = a(n)c + a(n+1), \quad (3)$$

$$[c_1, \dots, c_r] = 2[c_1 - 1, c_2, \dots, c_r] - 1, \quad c_1 > 0 \quad (4)$$

$$[c_1, \dots, c_r] = 1 + 2^{c_1}[c_2, \dots, c_r] \quad (5)$$

$$a([c_1, \dots, c_r] - 1) = a([c_2, \dots, c_r]) \quad (6)$$

$$a([c_1, \dots, c_r] + 1) = a([c_1 - 1, c_2, \dots, c_r]), \quad c_1 > 0 \quad (7)$$

Lemma 1 *Let c_i be a sequence of non-negative integers, and suppose that $c_2 > 0$. Then, for each $r \geq 2$,*

$$a([c_1, c_2, \dots, c_r]) = (c_1 + 1)a([c_2, c_3, \dots, c_r]) - a([c_3, \dots, c_r]). \quad (8)$$

Proof. Using (5), (7), and (3) with $c = c_1$, $n = [c_2, \dots, c_r]$, we find

$$a([c_1, \dots, c_r]) = c_1 a([c_2, \dots, c_r]) + a([c_2 - 1, c_3, \dots, c_r]). \quad (9)$$

Then using (6) and (3) again with $n = [c_2 - 1, c_3, \dots, c_r]$, we find

$$a([c_2, \dots, c_r]) = a([c_2 - 1, c_3, \dots, c_r]) + a([c_3, \dots, c_r]). \quad (10)$$

Comparing (9) and (10), the result follows. ■

Theorem 1 *Let $c_i : 1 \leq i \leq m$ be a sequence of positive integers. Then*

$$a([c_1, c_2, \dots, c_m]) = q_r(c_1 + 1, c_2 + 1, \dots, c_m + 1).$$

Proof. Define polynomials p_r by $p_0 = 1$, and $p_r(x_1, \dots, x_r) = q_r(x_1 + 1, \dots, x_r + 1)$. By Lemma (2) and Lemma (1), both $p_r(c_1, \dots, c_r)$ and $a([c_1, \dots, c_r])$ satisfy the polynomial recurrence relation

$$T_r(x_1, \dots, x_r) = (x_1 + 1)T_{r-1}(x_2, \dots, x_r) - T_{r-2}(x_3, \dots, x_r), \quad (11)$$

with initial conditions $T_0 = 1$, $T_1(x_1) = x_1 + 1$. ■

The expressions for $a([c_1, \dots, c_r])$ for the first few values of r are recorded below. The case $r = 3$ was recently derived by Defant in [2, eq (29)].

$$\begin{aligned} a([c_1]) &= c_1 + 1 \\ a([c_1, c_2]) &= c_1 + c_1 c_2 + c_2 \\ a([c_1, c_2, c_3]) &= c_1 c_2 c_3 + c_1 c_2 + c_1 c_3 + c_2 c_3 + c_2 - 1 \\ a([c_1, c_2, c_3, c_4]) &= c_1 c_2 c_3 c_4 + c_1 c_2 c_3 + c_2 c_3 c_4 + c_1 c_3 c_4 \\ &\quad + c_1 c_2 c_4 + c_2 c_4 + c_1 c_3 + c_2 c_3 - c_1 - c_4 - 1 \end{aligned}$$

The following corollary of the previous theorem corresponds to Corollary 3.3 of [3].

Corollary 1 *For $r \geq 1$, define the matrix*

$$M_r(y_1, \dots, y_r) = \begin{pmatrix} y_1 & 1 & 0 & 0 & 0 \\ 1 & y_2 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 1 & y_r \end{pmatrix}$$

Let $c_i : 1 \leq i \leq m$ be positive integers, and let I_r be the $r \times r$ identity matrix. Then

$$a([c_1, \dots, c_r]) = \det(I_r + M_r(c_1, \dots, c_r)).$$

Proof. It is easy to check that $\det(M_r(y_1, \dots, y_r))$ satisfies the same recurrence relation as $q_r(y_1, \dots, y_r)$, with the same initial conditions. So $\det(M_r(y_1, \dots, y_r)) = q_r(y_1, \dots, y_r)$. ■

While no easily discernible pattern is apparent in the polynomials p_r (or equivalently the expressions for $a([c_1, \dots, c_r])$ given above), listing the first few polynomials q_r reveals a surprising structure:

$$\begin{aligned} q_2(y_1, y_2) &= y_1 y_2 - 1 \\ q_3(y_1, y_2, y_3) &= y_1 y_2 y_3 - y_1 - y_3 \\ q_4(y_1, y_2, y_3, y_4) &= y_1 y_2 y_3 y_4 - y_1 y_2 - y_1 y_4 - y_3 y_4 \\ q_5(y_1, y_2, y_3, y_4, y_5) &= y_1 y_2 y_3 y_4 y_5 - y_1 y_2 y_3 - y_1 y_2 y_5 - y_1 y_4 y_5 - y_3 y_4 y_5 + y_1 + y_3 + y_5 \end{aligned}$$

The next theorem gives a precise description of this structure. For integers $r \geq 1$ and $1 \leq s \leq r$, define the sets

$$A_{r,s} = \{(i_1, i_2, \dots, i_s) : 1 \leq i_1 < i_2 < \dots < i_s \leq r, i_j \equiv j \pmod{2}\},$$

and $A_{r,0} = \{0\}$. So $A_{r,s}$ consists of increasing sequences of integers that start with an odd number and then alternate between even and odd numbers. For example, $A_{5,3} = \{(1, 2, 3), (1, 2, 5), (1, 4, 5), (3, 4, 5)\}$.

If $u = (i_1, i_2, \dots, i_s) \in A_{r,s}$, we write $y_u = y_{i_1} y_{i_2} \dots y_{i_s}$, and $y_0 = 1$. For $r \geq 1$ and $0 \leq s \leq r$, let $\omega_{r,s} = (-1)^r \cos(\pi(r+s)/2)$.

Theorem 2 *If $r \geq 2$, then*

$$q_r(y_1, y_2, \dots, y_r) = \sum_{s=0}^r \omega_{r,s} \sum_{u \in A_{r,s}} y_u. \quad (12)$$

Proof. We will show that the right side of (12) satisfies the recurrence (2). If $r \geq 1$ and $1 \leq s \leq r$, let

$$B_{r,s} = \{(i_1, i_2, \dots, i_s) \in A_{r,s} : i_1 = 1\},$$

$$C_{r,s} = \{(i_1, i_2, \dots, i_s) \in A_{r,s} : i_1 > 1\}.$$

Then $A_{r,s}$ is the disjoint union of $B_{r,s}$ and $C_{r,s}$. There are bijections $\phi : B_{r,s} \rightarrow A_{r-1,s-1}$ and $\psi : C_{r,s} \rightarrow A_{r-2,s}$ given by $\phi((1, i_2, \dots, i_s)) = (i_2 - 1, i_3 - 1, \dots, i_s - 1)$ and $\psi((i_1, i_2, \dots, i_s)) = (i_1 - 2, \dots, i_s - 2)$. If $z_i = y_{i+1}$, $1 \leq i \leq r-1$ and $w_i = y_{i+2}$, $1 \leq i \leq r-2$, then $y_u = y_1 z_{\phi(u)}$ for $u \in B_{r,s}$ and $y_u = w_{\psi(u)}$ for $u \in C_{r,s}$. The result then easily follows by splitting the sum over $A_{r,s}$ as $\sum_{u \in B_{r,s}} + \sum_{u \in C_{r,s}}$, and using the fact that $\omega_{r,s+1} = \omega_{r-1,s}$, and $\omega_{r,s} = -\omega_{r-2,s}$. ■

The following corollary is attributed to B. Reznick in [4] (see [5, Lemma 2.5]).

Corollary 2 *If n is a positive integer, let \bar{n} denote the integer obtained by reading the digits in the binary expansion of n in reverse order. Then $a(\bar{n}) = a(n)$.*

Proof. It is enough to consider the case n odd. Note that if $n = [c_1, \dots, c_m]$, then $\bar{n} = [c_m, \dots, c_1]$. So the result will follow if we show that $q_m(y_1, \dots, y_m) = q_m(y_m, \dots, y_1)$. This follows easily from either Corollary 1, by a permutation of the rows and columns that reverses the main diagonal of the matrix M_r , or from Theorem 2, by noticing that there is an involution $\beta_{r,s}$ on the sets $A_{r,s}$ given by $(i_1, \dots, i_s) \mapsto (i'_1, \dots, i'_s)$, where $i'_j = r - i_{s-j+1} + 1$, because if r and s have the same parity, then $r - i_{s-j+1} + 1 \equiv r - s + j \equiv j \pmod{2}$, while if $r \not\equiv s \pmod{2}$, then $\omega_{r,s} = 0$. ■

Proposition 3 *If $r \geq 0, k \geq 0$, then*

$$\begin{aligned} q_{k+r}(t_1, \dots, t_k, y_1, \dots, y_r) &= q_k(t_1, \dots, t_k)q_r(y_1, \dots, y_r) \\ &- q_{k-1}(t_1, \dots, t_{k-1})q_{r-1}(y_2, \dots, y_r). \end{aligned}$$

Proof. The proposition is proved by induction on k , by writing $q_{r+(k+1)} = q_{(r+1)+k}$, and making use of Corollary 2. ■

As a consequence of the last proposition, we obtain a simple proof of the following result of Coons [1].

Corollary 3 *If e, u and c are non-negative integers with $c \leq 2^e$, then*

$$a(c)a(2u+5) + a(2^e - c)a(2u+3) = a(2^e(u+2) + c) + a(2^e(u+1) + c).$$

Proof. The result holds for $c = 0$ trivially, and for $c = 1$ or $u = 0$ by using the basic identities for the Stern's sequence. Decreasing e if necessary, we may assume that c is odd and $c \geq 3$. So there is some $k \geq 2$ and integers c_1, \dots, c_{k-1} such that $c = [c_1, \dots, c_{k-1}]$. Since $c \leq 2^e$, we can define the positive integer $c_k = e - (c_1 + \dots + c_{k-1})$, and then $[c_1, \dots, c_k] = c + 2^e$. Write $u+2 = [u_1, \dots, u_r]$ for some positive integers u_1, \dots, u_r . Then $[c_1, \dots, c_k, u_1, \dots, u_r] = c + 2^e(u+2)$, and it is easily checked that $q_{r-1}(u_2+1, \dots, u_r+1) = a(u+1)$. Proposition 3, (with $y_i = u_i + 1, t_i = c_i + 1$, gives us the identity

$$a(c + 2^e(u+2)) + a(c)a(u+2) = a(c + 2^e)a(u+2) + a(c)a(u+3).$$

Use $a(2u+3) = a(u+2) + a(u+1)$, $a(2u+5) = a(u+2) + a(u+3)$, the basic identity $a(2^e + c) = a(2^e - c) + a(c)$ (see[4]) and the identity $a(2^e - c)a(u+1) + a(c)a(u+2) = a(2^e(u+1) + c)$ (easily proved by induction on e) to get the result. ■

The following result is an easy consequence of the recurrence (2) satisfied by the polynomials q_r . Recall that $s(n)$ is the number of 1's appearing in the binary expansion of n .

Corollary 4 *Suppose k is a positive integer that divides the exponent of each power of 2 appearing in the binary expansion of n . Then:*

$$a(n) \equiv \begin{cases} 0 \pmod{k} & \text{if } s(n) \equiv 0 \text{ or } 3 \pmod{6} \\ 1 \pmod{k} & \text{if } s(n) \equiv 1 \text{ or } 2 \pmod{6} \\ -1 \pmod{k} & \text{if } s(n) \equiv 4 \text{ or } 5 \pmod{6}. \end{cases}$$

Proof. We may assume that n is odd. Let $m = s(n)$. By assumption, $n = [kc_1, \dots, kc_m]$, and by Theorem 1, $a(n) = q_m(kc_1+1, \dots, kc_m+1) \equiv q_m(1, \dots, 1) \pmod{k}$. If $b_m = q_m(1, \dots, 1)$, then b_m satisfies the recurrence $b_m = b_{m-1} - b_{m-2}$ with initial conditions $b_0 = b_1 = 1$. This recurrence is easily solved as $b_m = \cos(m\pi/3) + \sin(m\pi/3)/\sqrt{3}$ and the result follows. ■

We conclude with a Binet type formula easily obtained by letting all variables of q_r equal a single variable t .

Corollary 5 *For all integers $r \geq 1$ and $t \geq 2$,*

$$a\left(\frac{2^{rt} - 1}{2^t - 1}\right) = \frac{\lambda^r - \mu^r}{\sqrt{(t-1)(t+3)}},$$

where

$$\lambda = \frac{t+1 + \sqrt{(t-1)(t+3)}}{2}, \quad \mu = \frac{t+1 - \sqrt{(t-1)(t+3)}}{2}.$$

Proof. Note that $(2^{rt} - 1)/(2^t - 1) = [t, \dots, t]$, where there are $r-1$ entries. So $b_r = a((2^{rt} - 1)/(2^t - 1)) = q_{r-1}(t+1, \dots, t+1)$ satisfies the recurrence $b_r = (t+1)b_{r-1} - b_{r-2}$ with initial conditions $b_1 = 1, b_2 = t+1$. Solving this recurrence we obtain the result. ■

Remark The previous corollary lends itself to natural generalizations, by considering for example $q_r(t, s, t, s, \dots)$ or $q_r(t, s, u, t, s, u, \dots)$ and so on. We leave the exploration of the corresponding formulas for the Stern's sequence to the interested reader.

Acknowledgment I thank Sam Northshield for pointing out the article [3] (that led to the current title of this note) and for several helpful comments on an earlier version of the paper, and Christophe Vignat for pointing out the article [1].

References

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